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THE QUANTUM INTEGRAL AND DIFFRACTION BY A CRYSTAL

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Duane has recently pointed out[1] that if the momentum of a crystal grating perpendicular to the crystal face is nh/a, where n is an integer, h is Planck's constant and a is the distance between successive atomic layers, and if the momentum of the incident radiation quantum is $h\nu/c$, then Bragg's diffraction formula $n\lambda = 2a \sin\theta$ is a necessary consequence. It is worth while to point out that the general statement of the quantum postulate, $\int p\, dq = nh + \eta$, leads directly to the result that the momentum of the crystal changes by integral multiples of h/a as Duane assumes.

Let us express our quantum postulate in the form

$$\bar{p} \equiv \frac{\int p\, dq}{\int dq} = \frac{nh}{q_1} + \gamma, \qquad (1)$$

where $\bar{p}$ is the displacement average of the momentum, $q_1 = \int dq$ is the displacement necessary to bring the system back to its original condition, and the constant γ, corresponding for example to the zero point energy of an oscillator according to Planck's second radiation formula, represents the minimum value of the average momentum. Applying this expression to the case of a beam of infinite plane waves of wave-length λ, it is clear that after the beam has propagated itself through a complete wave-length it is again in its original condition. Thus $q_1 = \lambda$. The momentum of the beam in the direction of propagation is therefore

$$p = nh/\lambda + \gamma.$$

This corresponds, according to the relativity theory, to an energy

$$\epsilon = pc = nhc/\lambda + \gamma c = nh\nu + \gamma c.$$

If $n = 1$ and $\gamma = 0$, we thus have $\epsilon = h\nu$, which is in accord with the results of photoelectric experiments. Thus the momentum of the light ray is

$$p = h/\lambda. \qquad (2)$$

Let us consider in a similar manner the motion of an infinite three dimensional grating, such as a crystal. If a_x is the distance between the layers of atoms in the X-direction, the condition of the grating after it has moved a distance a_x is indistinguishable from its original condition, since layers of atoms again occupy the original positions. Thus in equation (1), $q_1 = a_x$, and hence

$$\bar{p}_x = n_x h/a_x + \gamma_x = p_x, \tag{3}$$

since $p_x = \bar{p}_x$ for uniform motion. Just as in the case of the light ray, where p was the momentum of the whole ray, so here p represents the momentum of the whole crystal. The constant γ_x in this equation assumes different values according to the motion of the axes relative to which the momentum of the crystal is measured. This equation states that the momentum of the crystal along the X-axis changes by integral multiples of h/a_x, i.e., that

$$\delta p_x = \frac{h}{a_x}\,\delta n_x, \tag{4a}$$

where δp_x is the change in the X component of the momentum, and δn_x is an integer. Similarly, if a_y and a_z are the distances between the layers of atoms along the Y and Z axes, respectively,

$$\delta p_y = \frac{h}{a_y}\,\delta n_y \quad \text{and} \quad \delta p_z = \frac{h}{a_z}\,\delta n_z. \tag{4b, c}$$

These expressions (4) ascribe a momentum to the crystal which is quantized in precisely the manner asumed by Duane. He had shown that the dimensions of the equations demanded a length in the denominator of the right-hand members, and the lengths a were the only ones which appeared suitable; but the constant of proportionality, unity, remained arbitrary. We now see that this quantized momentum of the crystal is a direct consequence of the fundamental quantum postulate.

We shall now proceed with the discussion of the passage of radiation through the crystal according to the method suggested by Duane, though in somewhat greater detail. Let l_x, l_y, l_z be the direction cosines of the incident ray, and l_x', l_y', l_z' the direction cosines of the diffracted ray. If λ is the wave-length of the incident ray and λ' that of the diffracted ray, the increase in the X component of the momentum of the ray by the diffraction is $hl_x'/\lambda' - hl_x/\lambda$. By the principle of the conservation of momentum, this change in the momentum of the radiation must be balanced by the change in the momentum of the crystal, i.e., $hl_x'/\lambda' - hl_x/\lambda + h\delta n_x/a_x = 0$. Thus

$$\frac{l_x'}{\lambda'} - \frac{l_x}{\lambda} + \frac{\delta n_x}{a_x} = 0, \quad \frac{l_y'}{\lambda'} - \frac{l_y}{\lambda} + \frac{\delta n_y}{a_y} = 0, \quad \frac{l_z'}{\lambda'} - \frac{l_z}{\lambda} + \frac{\delta n_z}{a_z} = 0. \tag{5}$$

The total change in momentum of the crystal is $(\delta p_x^2 + \delta p_y^2 + \delta p_z^2)^{1/2}$, or

$$\delta p = h \left\{ \left(\frac{\delta n_x}{a_x}\right)^2 + \left(\frac{\delta n_y}{a_y}\right)^2 + \left(\frac{\delta n_z}{a_z}\right)^2 \right\}^{1/2}. \qquad (6)$$

From the principle of the conservation of energy it follows that the change in energy of the radiation is balanced by the change in kinetic energy of the crystal. In order to avoid changes of wave-length due to the Doppler effect, we must suppose that the initial velocity of the crystal is zero, and hence also that the initial value of $p = 0$. The energy equation is then

$$\frac{hc}{\lambda'} - \frac{hc}{\lambda} + \frac{(\delta p)^2}{2M} = 0. \qquad (7)$$

Substituting the value of δp given in equation (6), and using in this equation the values of $\delta n/a$ given by expression (5), equation (7) becomes

$$\frac{hc}{\lambda} - \frac{hc}{\lambda'} = \frac{h^2}{2M} \left\{ \left(\frac{l_x}{\lambda} - \frac{l_x'}{\lambda'}\right)^2 + \left(\frac{l_y}{\lambda} - \frac{l_y'}{\lambda'}\right)^2 + \left(\frac{l_z}{\lambda} - \frac{l_z'}{\lambda'}\right)^2 \right\}. \qquad (8)$$

On multiplying both sides by λ/hc we obtain

$$1 - \frac{\lambda}{\lambda'} = \frac{h/\lambda c}{2M} \left\{ \left(l_x - l_x' \frac{\lambda}{\lambda'}\right)^2 + \left(l_y - l_y' \frac{\lambda}{\lambda'}\right)^2 + \left(l_z - l_z' \frac{\lambda}{\lambda'}\right)^2 \right\}. \qquad (9)$$

In this expression $h/\lambda c = h\nu/c^2$ is the mass of the incident quantum of radiation, which is very small indeed compared with the mass M of the crystal. Hence we have almost exactly

$$1 - \lambda/\lambda' = 0, \text{ or } \lambda' = \lambda. \qquad (10)$$

When the value of λ' given by equation (10) is substituted in equations (5), we obtain

$$l_x - l_x' = \delta n_x \frac{\lambda}{a_x}, \quad l_y - l_y' = \delta n_y \frac{\lambda}{a_y}, \quad l_z - l_z' = \delta n_z \frac{\lambda}{a_z}. \qquad (11a, b, c)$$

These expressions are exactly those obtained on the theory of interference for the angles at which the ray of wave-length λ may be diffracted by a crystal. Here, however, they are based upon the energy and momentum principles and the quantum postulate.

In order to put this result in the more familiar form known as Bragg's law, let us suppose that the incident ray lies in the XY plane, and that momentum is imparted only along the X axis, i.e., we assume $l_z = 0$, $\delta n_y = 0$ and $\delta n_z = 0$. It follows from equation (11c) that $l_z' = 0$, meaning that the diffracted ray also lies in the XY plane. If we call θ the glancing angle of incidence of the ray as it strikes the YZ plane of the crystal

and θ' the glancing angle of emergence, we have $l_x = \sin\theta$, $l_x' = -\sin\theta'$, $l_y = \cos\theta$ and $l_y' = \cos\theta'$. Equation (11b) thus becomes

$$\cos\theta - \cos\theta' = 0,$$

whence $\theta' = \pm\theta$. If $\theta' = -\theta$, we find from (11a) that $\delta n_x = 0$, whence if the ray is undeflected no momentum is imparted to the crystal. If, however, a number $\delta n_x = n$ quanta of momentum are imparted to the crystal, we must have $\theta_1 = \theta$, and equation (11a) becomes

$$n\lambda = 2a_x \sin\theta. \qquad (12)$$

This is identical with Bragg's expression, derived from the usual interference considerations, in which n represents the order to the diffracted beam.

It will be noted that this derivation of equations (11) and (12) has assumed infinitely long trains of waves, and a diffracting crystal which is infinite in extent. The same assumptions are also used (at least implicitly) when these expressions are derived on the interference theory. In both cases the modifications for finite wave-trains and finite crystals may be made by considering these finite quantities as the Fourier integrals of infinite wave-trains or gratings. The equations thus resulting from the quantum postulate have been given by G. Breit;[2] though in accord with the viewpoint of the present paper, we should consider the momentum of the crystal itself to be quantized rather than Breit's suggestion of some disturbance traversing the crystal.

The argument leading to equation (9) is precisely similar to that used by the writer in calculating the change in wave-length when X-rays are scattered by individual electrons,[3] except that in the latter case the mass $h/\lambda c$ of the radiation is comparable with the mass m of the scattering electron, so that the change in wave-length becomes appreciable. The fact that equation (9) indicates no measurable change in wave-length for the diffracted ray suggests that it is scattering by large groups of electrons, such as atoms or minute crystals, which gives rise to the scattered X-ray of unmodified wave-length.[4]

Attention may well be called to the fact that the present quantum conception of diffraction is far from being in conflict with the wave theory. In fact we were able to quantize the incident radiation only in view of the fact that it repeats itself at regular space intervals. Thus even from the quantum viewpoint electromagnetic radiation is seen to consist of waves.

[1] W. Duane, *Proc. Nat. Acad. Sci.*, May, 1923.
[2] G. Breit, *Ibid.*, July, 1923.
[3] A. H. Compton, *Physic. Rev.*, May, 1923.
[4] A. H. Compton, *Ibid.*, July, 1923; P. A. Ross, *Proc. Nat. Acad. Sci.*, July, 1923.

THE PRINCIPLE OF MINIMUM ENERGY AND THE MOTION OF FLUIDS

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The discussion will be limited to the case of an incompressible fluid, filling completely a simply connected region enclosed between boundaries, which are subject to certain prescribed motions. The fluid is acted on by no forces except those due to the movements of the boundaries and is initially at rest.

In establishing the fundamental equations for the kinematical relations it is usual to assume "irrotational" and "continuous" motion as fundamental conditions. The former leads at once to the existence of a velocity potential and the latter to fulfilment of the boundary conditions.

Lord Kelvin has shown that "The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal motion of the boundaries."[1] The proof of this theorem was first published in 1849,[2] and is based on a comparison of the motion controlled by a velocity potential with that of any other motion which is kinematically possible and consistent with the prescribed boundary conditions.

Lord Kelvin has shown[3] also that on the assumption of minimum kinetic energy an impulsive change of shape of the boundary surface will produce motion of the non-rotational character provided the kinematical condition of continuity is fulfilled.

It is the object of this paper to show that if the condition of minimum energy is to be fulfilled, the motion must necessarily be irrotational and the equation of continuity must be satisfied and it is then argued that since, under certain specified conditions, these requirements are with necessity fulfilled and may be postulated *a priori*, the motion must under those circumstances always be one of minimum energy. Thus the Principle of Minimum Energy for an incompressible fluid is established by a direct and simple line of reasoning.

We shall begin by investigating what the kinematical relations must be if the kinetic energy is to be a minimum.

Motion under the Condition of Minimum Energy.—Consider a mass of fluid enclosed between an inner closed bounding surface of any shape and an outer bounding surface which may be at a finite distance from the inner or at infinity. The case of a single closed surface is therewith included. The surfaces will be referred to as (1) and (2), respectively, and one or both of them are supposed to have a prescribed motion.

Let u, v, w be the component velocities of a fluid element parallel to rectangular coördinate axes at the point (x, y, z) at a given instant. We make no assumption as to these velocities except that they are each of them functions of (x, y, z) and the time t, but we know nothing of the form of the funtcions or of any relation between them.

Since we consider the motion kinematically such as it is at a particular instant, we may in this connection regard t as a constant. No assumption is made *a priori* as to the equation of continuity or as to the motion being irrotational. It is not even necessary to assume that the fluid is "perfect."

The kinetic energy is given by the equation

$$T = {}^1/_2 \rho \int\int\int (u^2 + v^2 + w^2) dxdydz \qquad (1)$$

where ρ is the density, here considered constant and uniform. In order that T shall be a minimum, the velocities u, v, and w must be functions of such a form, that when that form is made to vary arbitrarily by a small amount, the variation in energy, δT, shall be equal to zero. By the rules of Calculus of Variation, it is found that in order to fulfil this condition we must have:

$$u\delta u + v\delta v + w\delta w = 0 \qquad (2)$$

It is clear that if u, v and w were entirely independent of each other it would be impossible to satisfy this equation for all arbitrary variations of these quantities. It is concluded that u, v and w must be in some way derived from a common function and the first problem to be solved is to determine the relation between the component velocities and this common function, which we shall call ϕ.

Let the relations be expressed thus:[4]

$$u = f_1(\phi), \quad v = f_2(\phi), \quad w = f_3(\phi) \qquad (3)$$

where the f's are operators, the nature of which is yet unknown, and where ϕ is a function of (x, y, z) and a parameter. The function ϕ must be such that any small variation of its form will cause such variations in the quantities u, v and w as will make T a minimum and hence $\delta T = 0$.

From the equations (3) it follows that u, v, w are connected by some functional relation with x, y, z. We may therefore write:

$$F(u, v, w, x, y, z, a) = 0 \qquad (4)$$

where u, v, w are dependent variables and a is a constant. We have then to make the triple integral in (1) a minimum for all such variations in u, v and w as are consistent with (4). Following the method of variation we find

$$u\delta u + v\delta v + w\delta w + \lambda\left(\frac{dF}{du}\,\delta u + \frac{dF}{dv}\,\delta v + \frac{dF}{dw}\,\delta w\right) = 0$$

where λ is an indeterminate constant and dF/du, dF/dv, dF/dw are partial differential coefficients of F.

In order that this shall be true for all values of the variations we must have:

$$u + \lambda\, dF/du = 0, \quad v + \lambda\, dF/dv = 0, \quad w + \lambda\, dF/dw = 0$$

from which, by elimination of λ,

$$u : v : w = dF/du : dF/dv : dF/dw. \tag{5}$$

If for any given point, that is, for fixed values of x, y and z, we regard u, v, w as independent variables, equation (4) may be interpreted geometrically by considering u, v, w as coördinates in a rectangular system. In that system equation (5) represents a straight line drawn through the origin to the point (u, v, w) on the surface F, and since it is everywhere normal to F, that surface must be a sphere:

$$u^2 + v^2 + w^2 = r^2 \tag{6}$$

where r is the radius of the sphere and is equal to the resultant velocity. Equation (4) may now be written:

$$F[(u^2 + v^2 + w^2), x, y, z, a] = 0 \tag{7}$$

which expresses the fact that the kinetic energy at any point of the fluid is a function of (x, y, z). The sphere defined by equation (6) will be referred to as the "sphere of uniform energy" or simply the "energy sphere."

The relations here determined hold good throughout the region. According to (5) the component velocities at any point (x, y, z) are proportional to the partial differential coefficients of some surface, and it follows that the operators f in equations (3) stand for partial differentiations. Hence, if we make

$$u = d\phi/dx, \quad v = d\phi/dy, \quad w = d\phi/dz \tag{8}$$

we shall be in accord with equations (5) and satisfy the condition implied in (3) that the component velocities are derivable from the same function. This function may be written:

$$\phi(x, y, z) = c \tag{9}$$

where c is a parameter which, when given different values, defines a family of surfaces ϕ in the x, y, z system. If the coördinate axes are fixed in space and one of the boundaries is in motion relative to the other, c must be a function of time. The function ϕ is known as a "*velocity potential*," the existence of which is thus shown to be a necessary condition if the

kinetic energy of the fluid is to be a minimum. By partial differentiation of the equations (8) we obtain the relations:

$$dw/dy - dv/dz = 0, \quad du/dz - dw/dx = 0, \quad dv/dx - du/dy = 0 \quad (10)$$

where the quantities on the left-hand side express the rotation of an element of the fluid about axes parallel with the coördinate axes. Since these quantities are everywhere zero, we arrive at the important result that for minimum energy the motion must be "irrotational," and it follows that *the fluid must be frictionless or "perfect."*

Insert now the values of u, v, w from (8) in (1) and we obtain:

$$T = {}^1/_2 \rho \int\int\int \{(d\phi/dx)^2 + (d\phi/dy)^2 + (d\phi/dz)^2\} dxdydz \quad (11)$$

The function ϕ is still unknown, but must be such that when it is given an arbitrary variation, $\delta\phi$, in its form, the variation δT in the integral shall be zero. By the method of variation we find that this requires

$$d^2\phi/dx^2 + d^2\phi/dy^2 + d^2\phi/dz^2 = 0 \quad (12)$$

which is Laplace's equation, in this connection known as the "*equation of continuity.*" The integration of (12) furnishes the solution, subject, however, to the conditions existing at the boundaries.

It remains then to determine the motion at the boundaries, and to this end we now consider the terms at the limits of the integral obtained by the variation of T. The Calculus of Variation furnishes two equations, identical in form, one applying to the inner boundary (1), the other to the outer boundary (2):

$$(d\phi/dz - dz/dx\, d\phi/dx - dz/dy\, d\phi/dy)_{1,2} = 0 \quad (13)$$

where dz/dx and dz/dy are obtained from the given equations of the bounding surfaces:

$$F_1(x, y, z, t) = 0 \text{ and } F_2(x, y, z, t) = 0 \quad (14)$$

Equation (13) may be given the form for surface (1):

$$(u\, dF_1/dx + v\, dF_1/dy + w\, dF_1/dz)_1 = 0 \quad (15)$$

where dF_1/dx, etc., are the direction cosines of the normal to surface (1).

This equation expresses the fact that the normal velocity relative to the surface shall be zero. If now the surface is in motion, there must be added to the terms inside brackets in (15) an expression giving the velocity of the surface at the point under consideration with negative sign. This is the mathematical requirement that the terms at the limits of integra-

tion shall be zero. The equation then takes the general form

$$(u\,dF_1/dx + v\,dF_1/dy + w\,dF_1/dz - dF_1/dt)_1 = 0 \qquad (16)$$

which must be satisfied by the component velocities at any point infinitely near to the boundary. A similar equation will be obtained for the bounding surface (2) if it is in motion. If the latter surface is at infinity at all points and the fluid is at rest at infinity, all terms of the equations (16) vanish for that boundary.

It is seen that the fluid forms a closed kinematical chain. If we prescribe a certain motion for some particles, all the others have to move in a definite orderly manner, and by a purely mathematical reasoning we have found what this motion must be.

Proof that the Motion Must Be One of Minimum Energy.—It has been shown that if the condition of minimum energy is to be satisfied, the motion must comply with the following four kinematical requirements:

1. It must be regulated by a velocity potential.
2. It must be irrotational, which follows as a corollary from (1) and entails that the fluid must be perfect.
3. It must satisfy the equation of continuity.
4. It must satisfy the boundary conditions.

Let us now examine under what conditions these requirements are fulfilled.

Condition (4) is satisfied when the velocity of the fluid normal to the boundaries is zero at every point of the bounding surfaces, and this is obviously the case where those surfaces are impenetrable to the fluid.

Condition (3), for an incompressible fluid, expresses the requirement that the amount of fluid which enters a space element in a small time dt is equal to that which flows out of it. This requirement might be violated if the particles of fluid at some point not contiguous with the boundaries separated from each other and left between them a vacant space or a space filled with vapors. It is therefore necessary to assume that the fluid fills the space between the boundaries completely, for in that case such discontinuities cannot occur in an incompressible fluid, so long as each molecule occupies the same volume. But it may happen that the molecules penetrate into one another as they do in chemical reactions or that the temperature changes in an uneven manner in both of which cases the volume of the molecules will be altered and the equation of continuity will not always be satisfied. Such conditions must therefore be excluded from consideration.

Condition (2) requires that the motion shall be irrotational. This is necessarily the case in a frictionless fluid under the fundamental conditions here assumed, where the motion is created from rest entirely by movements of the boundaries, for molecular rotation cannot be created by normal

pressures. If the fluid is not frictionless, the motion will not necessarily or in general be irrotational, hence it follows that if we want to insure that the motion shall be irrotational we must assume that the fluid as a whole—or at least the part which we consider—is "perfect."

Condition (1) is fulfilled if condition (2) is. With irrotational motion a velocity potential is mathematically necessary, for in that case we have:

$$u(dv/dz - dw/dy) + v(dw/dx - du/dz) + w(du/dy - dv/dx) = 0$$

which is the condition that $udx + vdy + wdz$ shall be a perfect differential of some function $\phi(x, y, z)$.

Thus by retracing the steps in the reverse order, it is found that in a perfect incompressible fluid, subject to no chemical or thermal changes and filling completely a singly connected region enclosed between impenetrable but movable boundaries, all the conditions for minimum energy are satisfied. Hence the Principle of Minimum Energy is therewith established under those conditions.

The Surface of Uniform Energy.—It is of interest to study in some detail the geometrical relations which exist between the sphere of uniform energy, equation (6), and the equipotential surfaces.

To every point on a certain energy sphere in the (u, v, w) system of coördinates, there corresponds a point in the (x, y, z) system. When the form of the velocity potential is known, the coördinates of such points are readily found, for let the point in the former system be (u_1, v_1, w_1), we have then:

$$(d\phi/dx)_1 = u_1 \quad (d\phi/dy)_1 = v_1 \quad (d\phi/dz)_1 = w_1$$

which give the values of (x_1, y_1, z_1) in terms of (u_1, v_1, w_1). Substituting these values in the equation for ϕ, we obtain the equipotential surface through that point: $\phi = \phi_1$.

The locus for all such points as (x_1, y_1, z_1) corresponding to a given energy sphere is the surface of uniform energy in the (x, y, z) system. It is found by substituting the values of the partial derivatives in

$$(d\phi/dx)^2 + (d\phi/dy)^2 + (d\phi/dz)^2 = r^2 \qquad (18)$$

which is the differential equation to that surface. When the velocity potential is known, this operation is a simple matter.

If in (18) we make r—the resultant velocity—a dependent variable, we obtain the relation denoted by F in equation (7), when $(u^2 + v^2 + w^2)$ is substituted for r^2 in (18).

In illustration consider the case of a circular cylinder of infinite length and of radius a, moving through an infinite mass of fluid with a velocity, u, in a direction normal to

its axis. The velocity potential is known to be:

$$\phi = a^2u_0x/(x^2 + y^2) \tag{19}$$

$$d\phi/dx = -u_0(y^2 - x^2)/a^2 \text{ and } d\phi/dy = 2u_0xy/a^2$$

The sphere of uniform velocity here becomes a cylinder, the equation of which is

$$u^2 + v^2 = r^2 \tag{20}$$

where r is an arbitrary, but constant velocity.

Corresponding to any point (u_1, v_1) on the cylinder (20), there is a point (x_1, y_1) in space determined from

$$u_1 = -u_0(y_1^2 - x_1^2)/a^2 \text{ and } v_1 = 2u_0x_1y_1/a^2$$

from which

$$y_1 = -(u_1 \pm r)x_1/v_1. \tag{21}$$

Hence corresponding to any one point on the cylinder of uniform energy there are two points in space where the resultant velocity is the same in magnitude and direction.

The surface of uniform energy in space is found by substitution in (18), which in this case leads to:

$$(x^2 + y^2)^2 = r^2a^4/u_0^2 \text{ or } x^2 + y^2 = ra^2/u_0^2 \tag{22}$$

which is the equation to a cylinder of radius $a(r/u_0)^{1/2}$. When $r = u_0$ this surface coincides with the boundary of the moving cylinder, showing that the resultant velocity at that boundary is everywhere equal to the velocity of the cylinder. Bearing in mind that $r^2 = u^2 + v^2$, it is seen that if we regard r as a dependent variable, the equation (22) corresponds to the function F of equation (7) and represents in the present case a paraboloid of revolution.

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[1] Horace Lamb, *Hydrodynamics*, **1916**, p. 45, where Lord Kelvin's proof of this theorem is given in substance.

[2] *Cambr. and Dubl. Math. J.*, "On the Vis-Viva of a Liquid in Motion."

[3] Thomson and Tait, *Natural Philosophy*, pp. 299–300.

[4] It is clear that the independent variables (x, y, z) cannot occur explicitly along with ϕ under the sign f, since in that case an independent variation of ϕ, which satisfies (2) at a given point, could not satisfy it at all other points. In other words, only ϕ itself, eventually with a constant added, can occur under f. It appears also that f cannot indicate one of the ordinary operations of arithmetic and algebra, since in that case equation (2) would not be satisfied by all variations of ϕ. It is not necessary at present to speculate further on the nature of f, which will be clearly defined by the solution following.

[5] As a simple illustration suppose the inner bounding surface is advancing through the fluid with component velocities u_1, v_1, w_1; the equation (16) then becomes:

$$(u-u_1)dF_1/dX + (v-v_1)dF_1/dy + (w-w_1)dF_1/dz = 0 \tag{17}$$

THE VOLUME CHANGES OF FIVE GASES UNDER HIGH PRESSURES

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This note briefly summarizes the most important numerical results of an investigation of the effect of pressure on the volume of hydrogen, helium, ammonia, nitrogen, and argon, which will be described in detail in a forthcoming number of the Proceedings of the American Academy of Arts and Sciences. The pressure range of this work reached a maximum of 15,000 kg./cm.^{2} for three of the five gases; the maximum pressure of previous measurements is 3000 by Amagat[1] for hydrogen and nitrogen of the five gases above. The temperature of most of this work is near 60°. Attempts to measure the thermal expansion under pressure gave results of less accuracy than the pressure changes listed here; these expansions will be found in the detailed paper.

The general method adopted is the same as that which I have previously used in measuring the compressibility of liquids.[2] The gas under an initial pressure of 2000 kg./cm.^{2} is placed in a pressure cylinder filled with kerosene. Pressure is produced in the kerosene and the gas by a piston forced into the cylinder, and the change of volume of gas and kerosene is determined by the motion of the piston. Blank runs allow the elimination of the effect of the compressibility of the kerosene and the elastic distortion of the cylinder. The volume changes of the gas found in this way are the changes from the initial pressure as zero. Various friction effects made it desirable to choose the fiducial zero pressure as 3000 instead of 2000 kg./cm.^{2} The actual volume in cc. per gm. or cc. per mol cannot be found from the measurements recorded here, but the values found by other methods and other observers at some pressure within the range of the present work must be employed to calculate from the observed changes of volume of this work the actual volumes.

The observed changes of volume in cc. per mol reckoned from 3000 kg./cm.^{2} as the zero are listed in the table, in which are also given the actual volumes of those three gases for which there is a basis for the calculation. The fiducial volumes at 3000 have been directly determined by Amagat[1] for hydrogen and nitrogen. The fiducial volume used for helium I have calculated from an equation of Keyes[3] which fits all previous experimental values over a small pressure range very well; the volume given by this equation at 3000 is doubtless somewhat too high. For the other two gases, ammonia and argon, not only are there no measurements of the vol-

ume at high pressures, but there are no published equations of state which can be admitted with any probability to give the high pressure volumes.

THE EFFECT OF PRESSURE ON THE VOLUME IN CC. PER MOL OF FIVE GASES

PRESSURE KG./CM.²	HYDROGEN AT 65°		HELIUM AT 65°		NITROGEN AT 68°		AMMONIA AT 30°	ARGON AT 55°
	ΔV	V	ΔV	V	ΔV	V	ΔV	ΔV
2000	...	....	...	....	...	....	−3.70	−3.31
3000	0.00	24.53	0.00	22.16	0.00	36.13	0.00	0.00
4000	2.29	22.24	3.08	19.08	2.49	33.64	2.04	1.96
5000	3.79	20.74	4.92	17.24	4.25	31.88	3.41	3.39
6000	4.91	19.62	6.16	16.00	5.52	30.61	4.44	4.47
7000	5.80	18.73	7.08	15.08	6.56	29.57	5.28	5.34
8000	6.47	18.06	7.80	14.36	7.40	28.73	5.93	6.06
9000	6.97	17.56	8.40	13.76	8.04	28.09	6.48	6.67
10000	7.41	17.12	8.89	13.27	8.61	27.52	6.97	7.18
11000	7.82	16.71	9.32	12.84	9.14	26.99	7.42	7.58
12000	8.16	16.37	9.64	12.52	9.60	26.53	7.85	8.00
13000	8.48	16.05	9.92	12.24	10.00	26.13	...	8.34
14000	...	....	10.20	11.96	10.36	25.77	...	8.66
15000	...	....	10.40	11.76	10.70	25.43	...	8.94

The volumes of hydrogen, helium and nitrogen run in the order helium, hydrogen, nitrogen at all pressures within this range. This is what would be expected from the molecular structure, the volume being most compressed for that substance with monatomic molecules, and least compressed for the molecules composed of large atoms. However, the complexity of structure of nitrogen results in a persistence of the compressibility at high pressures, as may be seen from the table which shows that the actual decrease of volume of nitrogen between 14,000 and 15,000 is 1.7 fold greater than the decrease of helium. This is naturally to be interpreted as due to the compression of the more complicated molecular structure itself. The same phenomenon will be found if the ΔV shown in the table is plotted against pressure. The curve for nitrogen crosses that of helium, and also the curve for argon crosses first that of ammonia and it is probable that it will also cross that for hydrogen at pressures slightly beyond those reached here. Again it is the more complicated structure which retains its compressibility at the highest pressures.

The actual compressibility of these gases under pressure, defined as $1/v.(\partial v/\partial p)_{,}$, is of the same order as the compressibility of normal liquids at pressures a few thousand kilograms lower.

The densities under 15,000 kg. are as follows: hydrogen, 0.1301 (extrapolated from 13,000), helium 0.340, and nitrogen 1.102. These densities are greater than the densities of the liquid or solid phases of these substances under reduced temperatures at atmospheric pressure. Thus the density of liquid hydrogen is ordinarily given as 0.070, and that of the solid as

0.076; the density of liquid helium is 0.1456; and that of liquid nitrogen is in the neighborhood of 0.85. These figures enable lower limits to be placed on the compressibility of the liquid and solid phases, quantities for which we previously have had no idea of the order of magnitude. It appears that the compressibilities of solid and liquid hydrogen and of liquid helium are by far the greatest ever found for a solid or liquid, and it is probable that these are the most compressible liquids and solids. Under 15000 kg. solid hydrogen is compressed to less (and probably much less) than 0.58 of its initial volume, and of course the liquid is proportionally more compressed. The extreme solid hitherto measured is metallic potassium[4] which under 15,000 would be decreased to 0.70 of its initial volume. Similarly liquid helium is compressed by 15,000 kg. to less (and probably much less) than 0.43 of its volume under atmospheric pressure. The extreme liquid hitherto measured is ether, which under 15,000 kg. (extrapolated from 12,000) is compressed to about 0.68 of its initial volume.

None of the published equations of state which I have been able to examine reproduce the behavior of these gases under high pressures. Most of the familiar equations predict a too rapid falling off of compressibility at high pressures; this is to be ascribed to failure to take into account the compressibility of the molecules. The most successful high pressure equation which has come to my attention is that recently proposed by Becker[5] for nitrogen. This is based on Amagat's data to 3000 kg. Used as an extrapolation equation beyond this pressure it predicts at 15,000 kg./cm.^{2} a molar volume of 25.0 against 25.4 found experimentally.

[1] Amagat, E. H., *Ann. Chim. Phys. Paris* (6), **29**, 1893 (1–140).
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ACHROMATIC AND SUPERCHROMATIC FRINGES WITH A CALCITE RHOMB*

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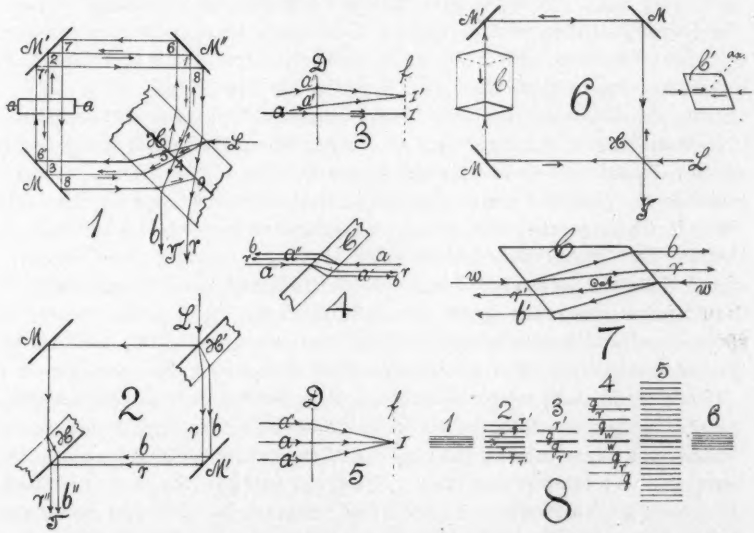
1. *Plate Glass Compensators.*—In the case of the Michelson interferometer, the fundamental fringe figure is elliptical, and the outer rings when the disc vanishes are progressively colored spectral rings. The superchromatics are not producible. One may note that even horizontally, a ray of white light comes out as a ray of white light.



In case of the modified interferometers if compensated, the displaceable superchromatic fringes are easily obtained with white light. The fundamental design may here be taken to consist of equilateral hyperbolas, with the horizontal black fringe as one of the asymptotes, though the figure is usually much more complicated. This does not necessarily interfere with the occurrence of available small straight fringes (cf. Carnegie Publications No. 249, part 4, chap. 7, par. 54, 55). In these adjustments, one may note, an entering ray of white light issues with horizontal separation of colors. Thus in the self-adjusting interferometer, figure 1, with or without the compensator (the latter is here superfluous except as a film protection) the incipient white ray L is replaced by the

spectrum b, r. In the quadratic interferometer, figure 2, in spite of the complete compensation in view of the identical half silvers H and H', the entering white ray L is again changed to the horizontal spectrum $r''b''$. There must therefore in these cases also be vertical spectra with inclined rays.

The superchromatic fringes are in appearance like Young-Fresnel fringes, except that the central fringe is often not white but black. In other words, there may be phase reversal corresponding to the black spot in thin plates.

If for convenience we operate with the self-adjusting interferometer, figure 1, and rotate the mirror M slightly about the horizontal axis, the fringes may be made to enlarge from horizontal hair lines, through infinite size back again to hair-lines for the same sign of rotation. The coincident slit images rise or fall in the telescope; at the same time the black fringe is changed in position correspondingly. The whole phenomenon as to color, etc., passes through infinity symmetrically with respect to the black line in spite of the marked change in angle of the rays.

Since the distances 34 and 3214 are unequal, it is obvious that the spots of light at 4 move vertically over each other when M is rotated on a horizontal axis, although the rays remain parallel. Hence let D, figure 3 (elevation), be the principal plane of the objective of the telescope at T and let the coincident parallel rays a have an image at I on the focal plane f; then in case of rotation of M, the coincident rays a separate into the rays a', a'' with their focus at I', again coincident; but whereas the interference design from contiguous spots, a, is infinite in size, that of the separated spots is reduced proportionally to their distance apart. The position of the black line of symmetry will depend on the phases of a' and a''.

If the equidistant mirror M' (figure 1) is rotated on a horizontal axis, the slit images move in the telescope as before, owing to the increasing obliquity of rays; but the fringes do not change in size because the distance $a'a''$ is relatively constant. M'' again operates like M. H rotated on a horizontal axis changes the size of fringes only, since but one component beam 41234 is reflected. Being reflected in parallel I remains in place, but a changes to $a'a''$ with a focus at I.

The other method of changing the size of fringes (with a particular bearing on the present experiment) consists in placing a compensator of good plate glass aa, figure 1, capable of rotating on a horizontal axis, in the path of the beams. In the oblique position, figure 4, this converts the originally coincident opposed rays a, a into the noncoincident but parallel rays $a'a''$. The result is, if D, figure 5, is the principal plane of the objective of the telescope at T (figure 1), that the coincident rays at a with an image at I, are converted into the symmetrically parallel rays $a'a''$ also with an image at I. The slit images thus remain fixed while the fringes decrease in size proportionally to the distance $a'a''$.

2. *Iceland Spar Compensator. Extraordinary Ray.*—With a plate glass compensator no matter how thick (glass columns up to 7 cm. were tried), the fringes remain appreciably the same on the two sides of the position for infinite size; i.e., they pass from hair fringes, through infinity, to hair fringes again, symmetrically when the compensator is rotated in a given direction. This is no longer the case when a calcite prism, C, figure 6, with the short diagonal vertical, is introduced. To bring out the phenomenon vividly, a calcite block 3 cm. thick is desirable. With a nicol, 7 cm. long, the results are correspondingly striking.

In dealing with polarized light it is well to observe at the outset that the ray at T having undergone successive reflections is itself very nearly polarized, with a plane of polarization horizontal. It is therefore necessary to make adjustments for vertical vibration. Furthermore if the long diagonal of C is vertical, the ordinary ray is in question and the crystal there behaves like common glass as it should.

If the short diagonal of C is vertical and the rotation of C on a horizontal axis is made from hair fringes on one side to hair fringes on the other, then on one side of the position for infinitely large fringes they are pronouncedly achromatic and very numerous, whereas on the other side of the infinite fringes, they are few and usually highly colored. In view of the thickness of C a very small angle of rotation is in question.

Using the nicol 7 cm. long, with the short diagonal vertical, the number of achromatic fringes had further increased (contrasting therefore with the glass column 7 cm. long and its doubly symmetrical fringes). They fill the field of the telescope from top to bottom. In figure 8, I have given a record observed at successive stages of rotation, beginning on the colored side, w denoting white, r red, g green, r' reddish, etc. The black fringe is stationary throughout the rotation. The ∞-fringes are, of course, too faint within the field of the telescope to be characterized. On the achromatic side (4, 5, 6) the fringes merely shut up with loss of color from large to small size. On the colored side (3, 2, 1) the few present are variegated, but also narrow down to hair lines as the given rotation proceeds. The total rotation downward from the level edge, from small fringes to small fringes will not exceed 5°, but the fringes remain visible within an interval of about 15°. The design 2 is most interesting, consisting of a strong black line flanked by two vividly white lines on either side, the remaining colors soon fading off.

3. *Remarks.*—In the treatment of achromatic fringes and thin plates, it is customary to use the equation of condition $\cos r/\lambda = \text{const.}$, where r is the angle of refraction of the ray within the thin plate (here air, $\mu = 1$) and λ the wave-length. Applying this to different types of interferential apparatus, the usual result is the occurrence of the critical color function[1] $\lambda d\mu/\mu d\lambda$. But it is not necessary to enter into this degree of detail here,

since in case of compensated interferometers it is obvious from figure 4, that for plate glass the corresponding red rays are nearer each other than the blue rays and that the distribution is symmetrical on the two sides of the vertical position of C. Thus the dispersion due to refraction is superposed on the dispersion due to interference, so that the superchromatics (more or less of the type of No. 2, figure 8, with but few fringes and therefore very useful as moving indexes in measurement) are obtained.

For the case of the calcite rhomb C, figure 7, the two spectra br above, $r'b'$, below, issuing from the corresponding white rays w, w', are opposite in distribution. The red rays are here nearer each other vertically than the blue rays. Now it is only an extraordinary ray that can pass through the oblique face collinearly. Hence when the rhomb is rotated on an axis, A, normal to the diagrams, so that $r'b'$ is above and rb below, the blue rays are nearer together than the red rays. In the latter case they would therefore counteract the dispersion due to interference so that achromatic spectra would result. In the former case, they are in the same sense as the interferences and accentuate the color effects. Thus if c is the vertical distance $a'a''$ in figure 5 (normal distance between r, r' or b, b' in figure 7), since a' and a'' are usually in opposed phases, the equation of a dark line is $n\lambda = cx/F$, where n is the ordinal number of the fringe at a distance, x, from the center I and F the focal length of the objective. If within a certain range, approximately, $c = k\lambda$ (blues closer together), where k is a mean constant, $n = kx/F$, free from color. In the other case we should have to write consistently $c = k/\lambda$ (reds closer together) and $n\lambda^2 = kx/F$, accentuating color.

* Advance note from a Report to the Carnegie Institute, Washington, D. C.

[1] μ here referring to the glass used.

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*VIBRATION OF THE AIR FILAMENT IN QUILL TUBES. SINGLE TELEPHONIC EXCITER**

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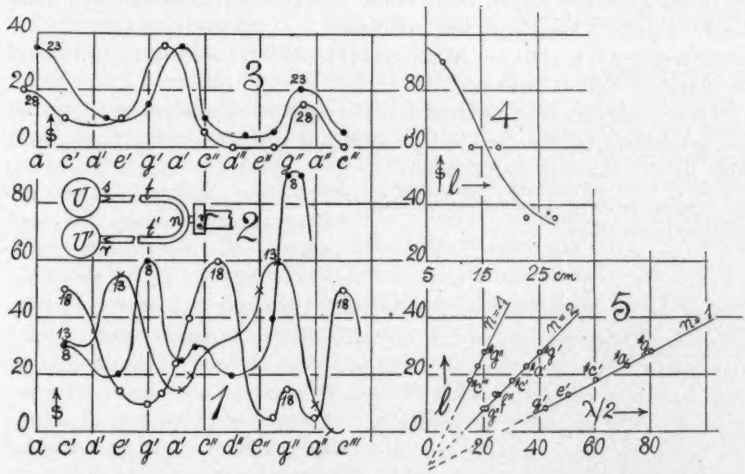
The preceding methods with two coöperating telephones introduce certain complicating inconveniences. In the case of straight tubes (longitudinal) the paired pin holes must be displaced to follow the nodes of the overtones. In transverse quill tubes the pin holes are fixed, but their tube length is a material addition to the remaining tube length. In this respect the adjustment given in figure 2 has advantages. But a single telephone, T, here actuates the paired quill pipes (.35 cm. in bore) sn and

rn, n being a node. With fixed pin holes r and s one should expect all the overtones to appear in succession; but for some reason the fifths formerly obtained are deleted. The results, however, are very much smoother than heretofore and three successive octave groups are available.

The survey in pitch is summarized in figures 1 and 3, the telephone, small inductor and circuit together constituting the electric siren. The lengths l are measured from s or r to n. Beginning with $l = 8$ cm. in the bend, l is increased in steps of 5 cm. on each side.

The chief maxima thus reappear at octave intervals. The optimum (see figure 4) lies within 8 cm. There are small secondary maxima which it is difficult to construe.



The relations of l and λ are given in figure 5, with the pitch added at each observation. Their disposition is again as nearly linear as the ear can guarantee, the three lines a, b, c belonging to the successive octave groups. Each again demands an initial tube length; but this (l = 12, 12, 13 cm.) is here nearly the same in the three cases, or is at least without systematic change. One might suppose that the tube ends Us and $U'r$ are implicated, but their length is but 5–6 cm. The equations of the lines for the frequency ratios n may be written

$$\begin{aligned} n = 1, \quad & l + 13 = .52(\lambda/2) \\ 2, \quad & l + 12 = 1.0(\lambda/2) \\ 4, \quad & l + 12 = 2.0(\lambda/2) \end{aligned}$$

The computed values, on the whole, are about as near as the ear may expect to come. Thus the viscosity expresses itself in the coefficient together with an additional tube length.

A question presents itself, whether these octaves (of which the middle one is the most conspicuous) are not stimulated by overtones in the telephone note. It is, however, improbable that the overtones should be of the observed intensity; and more important still, the telephone overtone can only be operative if the coefficient is fractional and $<.5$. Consequently we may write with assurance as in the preceding paper (if λ' is the wavelength under friction), taking the graph $n=1$ as the fundamental,

$$\begin{aligned} n = 1,\ \lambda' &= .52\lambda \\ 2,\ \lambda' &= .50\lambda \\ 4,\ \lambda' &= .50\lambda \end{aligned}$$

or in the general form $2(l + a) = b\lambda$. Since the constant a is here the same (nearly) throughout, the coefficient b is probably less in need of reduction in relation to a. At all events no observations could be devised to elucidate this dependence.

Thus it appears that the relations are simpler than heretofore, $b = .51$ being constant for the three octave groups $n = 1, 2, 4$ in frequency ratio. The values formerly found were:

For straight tubes of about the same section	$b = .51$	$a = 10$ cm.	$n = 1$
For transverse tubes.....	$b = .25$	$a = 2.5$	$n = 1$
	$b = .56$	$a = 22$	$n = 2$
	$b = .55$	$a = 22$	$n = 3$

These are all of the same order of value excepting the case $n = 1$ for transverse tubes, where $b = .25$ has but $^1/_2$ its anticipative value. Hence in the four octave groups $n = 1, 2, 3, 4$, b does not vary, if the exceptional case specified, associated with the exceptional $a = 2.5$ cm., can be explained. The endeavor to reduce b to b_0 for $a = 0$, by the method of the preceding paper, leads to no trustworthy results in the single telephone experiments.

A final question relative to the need of joining the pin holes s and r to the respective shanks U and U' of the gauge, deserves attention. In a final group of experiments, the pin hole r was therefore disconnected and open to the atmosphere. Another telephone was used; but the endeavor was made to insert the same tube length, $l = 8$ cm., as before. The graph thus obtained, was closely similar to the graph for $l = 8$ cm. in figure 1, a little shift necessarily resulting from the l-difference. It was astonishing to find a strong four-foot e in these short ($l = 8$ cm.) quill tubes and the whole curve above e, like figure 1, presented a case of multiresonance which only terminated at b''.

* Advance note from a Report to the Carnegie Institute of Washington, D. C.

[1] THESE PROCEEDINGS, 9, pp. 263–268, 1923, August.

PRELIMINARY ATTEMPT TO MEASURE GRAVIMETRICALLY THE DISTANCE-EFFECT OF CHEMICAL AFFINITY

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Chemical affinity is one of the most fundamental and important of the forces of Nature. It determines, of course, the existence of all chemical compounds, and therefore the possibility of life. Cohesive affinity, which may differ little in character from chemical affinity, is no less important, for it determines the condensation of all substances into liquids and solids. Liquefaction and solidification, too, are essential in building the complex machinery of the living organism.

Surprisingly enough, very little is certainly known of the true nature of these highly important forces. That they hold atoms and molecules together with great intensity, far exceeding that of gravitation, has been recognized for a long time. That they not only *hold* atoms together, but also cause atoms to *attract* one another, and therefore to exert pressure upon one another when in contact, is a conclusion which, although suspected by Humphry Davy[1] and a few others since his time,[2] has been supported by strong arguments only in the last quarter of a century.[3]

The question now arises as to the distance-effect of these very powerful forces. Every one agrees that their full effect is observable only at very short distances, and that their action is negligible at large distances. Even Sir Isaac Newton probably realized that the forces of cohesion and chemical affinity decrease more rapidly with increasing distance than does gravitation.[4] At the time of writing the *Principia*, however, his ideas upon this subject were vague, as is shown by the following quotation:

"And now we might add something concerning a certain most subtle Spirit, which pervades and lies hid in all gross bodies; by the force and action of which Spirit, the particles of bodies mutually attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles;...But these are things that cannot be explain'd in few words, nor are we furnish'd with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic Spirit operates."[5]

How searchingly Newton may have discussed the distance-effects of cohesion and chemical affinity in the papers burned in the disastrous fire which consumed so many years' work, no one knows. But there was not available during his lifetime, nor is there even now available, any "sufficiency of experiments" to decide the question indisputably.

Knowledge of the laws which regulate the decrease of chemical affinity and cohesion with increasing distance is no less important, as regards

comprehension of the universe, than is knowledge of the law of gravitation. Nevertheless, very little attention seems to have been directed towards their discovery. Few if any textbooks on chemistry or physics seem even to hint at their importance, or to suggest that they are worthy of experimental investigation.

Although direct experimentation has been infrequent, from time to time investigators have speculated and made assumptions bearing upon this subject. Young,[6] in the course of his studies on capillarity, appears to have thought that cohesive affinity is independent of distance "throughout the minute distance to which it extends" but that it ceases entirely at a distance of 1×10^{-7} mm. The experiments of Plateau, Quincke, Sohncke, Drude, Kelvin, Boltzmann and others were designed to show the range of potent action, or the size of the molecules, rather than a diminished effect at a distance.[7]

Yet other authors have assumed that cohesive affinity must vary inversely as some power of the distance—probably a higher power than that involved in the force of gravitation, which, of course, varies inversely as the square of the distance. Van der Waals[8] in 1873, in deducing his well-known and important equation for gases, assumed that cohesive affinity acts inversely as the square of the *volume* (or as the sixth power of the distance). The same assumption was later advocated by Pilling.[9] Seven years afterwards Eddy[10] contented himself with vaguer assumptions: that "some power of the distance" is concerned. In 1884 Harold Whiting[11] published his thesis upon a new theory of cohesion, and maintained that the true function is the fourth power of the distance. He admits that this is an assumption. In 1893 Richarz[12] attempted to prove by general theorems of mechanics that the general ideas involved do not afford sufficient stability and must therefore be rejected. More recently, in a long series of papers, Mills[13] assumes that the cohesion is inversely as the square of the distance—an assumption which might give cohesion a rôle in determining the motions of planets and satellites. Most of these investigations have confined their attention to cohesion, and have left chemical affinity entirely out of consideration.

Within the last few years, numerous theorists, stimulated by recent surprising discoveries and speculations concerning the structure of atoms, have usually at first assumed that, as regards the positive and negative electrical components of atomic structure, the law in question is primarily that of inverse squares. Since, however, searching analysis shows this assumption to be inadequate, many have pointed out that probably Coulomb's law may not hold at very small distances. Among these theorists may be mentioned especially Langmuir,[14] J. J. Thomson[15] and Born.[16] Most modern speculators agree with Newton in classing cohesive and chemical attractive affinities with electrical attraction, and endeavor to

compute the distance-effect from assumptions concerning the intimate internal electrical structure of the atoms. Many of the hypotheses are highly ingenious and interesting, but they can hardly be said to have yielded indisputable conclusions.

Considering the great importance of the subject, and the uncertainty of the hypothetical inferences which have thus far been available, every conceivable method of attack should be employed. Perhaps attempts have in the past been made to measure the distance-effect of the force of chemical affinity by actually measuring the attraction at finite distances, but these attempts, if any, are unknown to us. Two years ago, we made a few preliminary experiments in order to discover experimentally whether or not, on nearest possible approach, substances possessing strong chemical affinity for one another show any attraction which may be measured by an analytical balance. The experiments, although only qualitative, were definite in their outcome, and are perhaps worthy of brief notice.

The method consisted in arranging two substances, capable of reacting chemically upon one another, on a balance—supporting one of the substances (aluminium) on the balance beam, and another just beneath the first upon a bridge over the balance pan—in order to determine gravimetrically the existence or non-existence of attracting forces when the two surfaces were very close together.

A horizontal plate of aluminium, 6 cm. square, hanging from one arm of a balance, was exactly counterpoised; and in successive experiments this plate was allowed to rest very nearly upon the flat surfaces of a number of other substances with which the aluminium enters energetically into combination. The substances used for this purpose were, first, various oxides (of silver, copper, iron, zinc and magnesium), which show differing changes of free energy on reacting with aluminium, and afterwards sulfur, iodine and bromine. Each of these substances was placed upon an adjustable stand or bridge (of which the height and level were altered by suitable screws)—the final exact adjustment being made, partly by careful inspection and partly by the behavior of the balance itself, with the balance case closed.

Elaborate precautions were taken to avoid static electrical charges in the substances concerned, to exclude moisture, and to bring the surfaces of the two substances in each case as close together as possible. The powders were compressed into compact layers with very flat surfaces. Sulfur and iodine were cast upon plate glass to insure likewise flat surfaces, and the two halogens were prevented from attacking the aluminium by very thin plates of mica. This latter device unavoidably separated the surfaces by 0.02 mm. in the case of iodine and 0.1 mm. in the case of bromine, so that with these two substances much less satisfactory conditions were available than with the others. The oxides were in contact with the

aluminium plate over much of its surface, the distance between many particles of the two opposing substances being doubtless much less than 0.001 mm. Details of the experimental procedure will be given elsewhere.

In brief, the result was as follows: In no case, after taking all necessary precautions, was any attracting tendency observed capable of balancing as much as 0.1 mg. on the opposite arm.

Undoubtedly these experiments could be considerably improved. Although merely preliminary in nature, they demonstrate, nevertheless, that the force of chemical affinity (and therefore presumably also of cohesion) must fall off very rapidly with the distance, since it caused no appreciable effect in any of these cases, in some of which the free energy change involved was very large and the intervening distance very small. They are not, of course, able to afford any basis for the formulation of the exact law of this rapid decrease.

The negative outcome of our experiments is seen to be qualitatively in accord with the most plausible modern theories of the subject, drawn from the study of surface tension, elasticity and other properties of matter as well as from less direct evidence based upon hypotheses of atomic structure. Moreover, several other facts, well known to all chemists, point in the same direction. Highly unstable mixtures, such as potassium chlorate and sulfur, and the like, persist without change if undisturbed; but when compressed, even without raising the temperature, enter into reaction. The well-known experiments of van't Hoff and W. Spring,[17] continued by Spring alone, tended to show that reactions take place when solid substances concerned are pressed very closely together which will scarcely occur otherwise. The similar, more recent investigations of Johnston and Adams support this conclusion, with modifications as to detail.[18] The existence of such explosives as nitroglycerine and trinitrotoluene is corroborative evidence. Although the affinities concerned in such substances are not arranged in such a way as to have attained their most stable equilibrium, nevertheless the distance-effect even in these intimately associated atoms is too great to allow the attainment of the stablest position except under the influence of violent shock, which must compress at least some of the atoms so that the more powerful affinities may become operative.

Summary.—No evidence was found, by means of the balance, pointing toward an attractive force of as much as 0.1 mg. between a plate of aluminium and various substances upon which aluminium will react chemically, even on very close approach. Therefore it appears, as indeed has been often assumed, that the force of chemical affinity must decrease very rapidly as the distance between the attracting atoms increases.

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THE EFFECTS OF CUTTING THE GIANT FIBERS IN THE EARTHWORM, EISENIA FOETIDA (SAV.)

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In the course of some studies on the giant fibers in the nerve cord of the earthworm, Eisenia foetida (Sav.), I found it possible to cut these fibers and leave the major part of the cord intact. This afforded an opportunity of studying the reactions of a worm in which almost nothing but the giant fibers had been severed. Hence this method should give some evidence concerning the function of these fibers.

In a normal worm two definite reactions are well marked. One is the

ordinary creeping movement, and the other is the quick shortening of the animal as a whole. In a quiescent worm a slight stimulus applied to either end will cause coördinated creeping movements beginning at the opposite end and resulting in more or less progress in the direction of that end. A more vigorous stimulus at either end will cause the quick shortening of the whole worm.

Friedländer (1894) suggested upon anatomical grounds that the giant fibers had to do with the transmission of the impulse bringing about the quick shortening of the worm. Bovard (1918) studied the function of these fibers in worms undergoing regeneration. He took advantage of the difference in the rate of regeneration between the giant fibers and the short locomotor fibers. After complete severance of the cord the short fibers regenerate first, this regeneration being completed on the third or fourth day. About two days after the regeneration of the short fibers the giant fibers show physiological and anatomical continuity across the cut. Using worms during the interval between the regeneration of the two kinds of fibers, Bovard found that such worms crept in the normal way but that the quick contraction was altered in that it involved only that portion of the worm on the side of the operation to which the stimulus was applied. The part on the opposite side did not respond in the normal way. He interpreted this as meaning that the transmission of the impulse bringing about the complete contraction depended upon the giant fibers, which in this condition were not functioning across the cut. Work with drugs further confirmed this view.

I found it possible, using earthworms anaesthetized in 0.2% chloretone, to cut the giant fibers and leave the rest of the cord almost intact. This was done by making an incision about three segments long between the two pairs of chaetae on a side; the ventral body-wall was then bent down in such a way as to expose the dorsal part of the cord; and the giant fibers were cut without doing great harm to the rest of the cord. I made the incision about midway the length of the animal by thrusting a needle into the appropriate portion of the body-wall and pressing down upon it with a flat instrument. I was able to open the incision and handle the cord with small glass hooks. The giant fibers were easily visible in strong sunlight under a binocular dissecting microscope. I did the actual cutting of the fibers with a small steel needle sharpened to a cutting point on a hone and strop. I found it advantageous to sterilize the instruments by dipping them in strong alcohol.

The results of the operation were determined in three ways: by direct observation of the reactions of the worm after stimulation, by graphic records taken from living worms, and by examination of histological preparations.

After the most favorable operations the worm showed the ordinary creep-

ing reactions on the following day, quite as in unoperated individuals. But a sharp stimulus, which in normal worms brought about the quick shortening of the whole worm, induced a contraction that never passed beyond the wound but remained in that portion of the worm that was stimulated. The worms reacted in all respects like those that had not regenerated their giant fibers.

As a further check on the operation some of the worms were adjusted to an apparatus by which their contractions could be recorded on a kymograph. I used an outfit similar to the one described by Bovard. The worm was pinned through the operated region to a bit of cork fastened to a glass plate. A hook was attached to each end of the worm and connected with a writing lever. By this means I also demonstrated that the ordinary creeping movements extended through the whole length of the worm but that the quick shortening was limited to that section of the worm that was stimulated.

In a number of instances histological preparations were made of the region of the operation. Serial sections showed the giant fibers plainly, in front and behind the cut, with a varying gap of about 0.2 mm. where the cut was made. In the successful operations the ventral part of the cord including the chief short neurones appeared little disturbed, the cut involving apparently only the dorsal V-shaped area of connective tissue with its included giant fibers.

I obtained in one hundred and thirty-nine attempted operations nineteen cases in which the cutting of the giant fibers was clearly successful, without severing the other tracts to any great extent. Although this work is of a preliminary nature, it has progressed far enough to yield positive confirmatory evidence of the belief that the giant fibers are the paths of the impulses which bring about the quick contraction of the whole worm.

I wish to thank Dr. G. H. Parker for his kind interest and suggestions during the course of this work.

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ON THE RELATIVE VELOCITY OF BLUE AND YELLOW LIGHT

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The following discussion of certain astronomical observations appears to show conclusively that the velocities of blue and yellow light through interstellar space differ by less than one part in twenty thousand million. The test is differential and is quite independent of the actual velocity of light, which is as yet affected by errors probably as great as one part in twenty thousand.

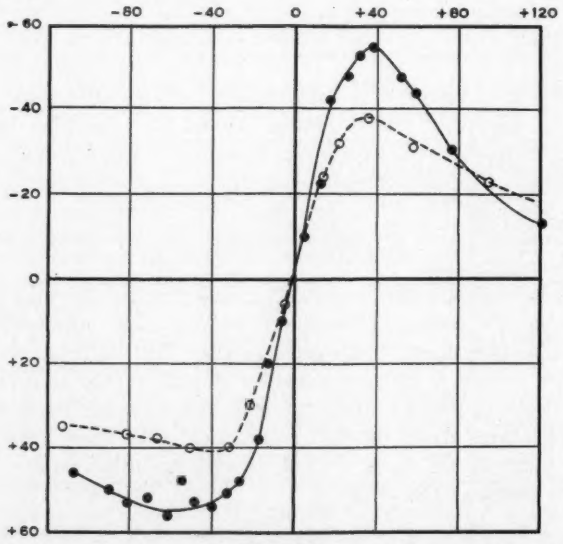

FIGURE I

Composite photographic (full line) and photovisual magnitude curves for cluster type variables in Messier 5. Ordinates are magnitudes in hundredths, and abscissae are phases in thousandths of a day.

For the accurate measurement of the relative velocity of light of different wave-lengths, there are three principal requisites—a source that emits vari-colored light signals at controlled or predictable intervals, a fairly accurately measured base line of great extent, and a device for recording the signals sent over this base line in light of different colors.

With the development of photometric methods of measuring great stellar distances, these requirements are now all fulfilled, I believe, in

the study of variable stars in globular clusters. The Cepheid variables of the cluster type provide the signals; the recently determined distance of the globular cluster Messier 5 gives a base line that is 2.3×10^{17} miles in length; and the ordinary photographic plates, used in a large telescope and with appropriate light filters, are satisfactory recorders of the transmitted light.

The Signaling Apparatus.—The periodic cluster type variable stars, of the subclass which shows very asymmetrical light curves and a total variation in brightness of a magnitude or more, are the most satisfactory for this investigation. The changing brightness can be timed with considerable certainty as it passes through any magnitude on the rising branch of the light curve. The maxima are usually sharp, and they too can be timed with some success; but the rapid rise is less dependent than the maximum on the exact form of light curve adopted. Observations with polarizing photometers, by Wendell at Harvard on the star RR Lyrae, and by the writer at Princeton on the similar variable SW Andromedae, showed, more than ten years ago, that the time of median magnitude can be determined within two or three minutes. Many of the variables of the cluster type rise from minimum to maximum, more than doubling the light emission of minimum, at the rate (as they go through median brightness) of one magnitude in thirty minutes.

In Messier 5 there are nearly one hundred cluster type variables, with periods ranging from six to twenty hours, and an average period of thirteen hours. In addition, this globular cluster contains a few long period Cepheids, which are of value in determining the parallax. The variability of nearly all these stars was discovered by Professor Bailey. He also investigated the periods and light curves.[1] For a few stars slight irregularity or variability in the period was found or suspected, but the changes are relatively small, and for the great majority it can be claimed that the periods are known within a fraction of a second. The time of a maximum, or of the passing through the median light on the ascending branch, can be predicted some years in advance with an uncertainty of only a few minutes.

The explosive character of the light variation of the cluster type variables is now generally interpreted as one result of pulsations in giant stars. But the use of these stars in the present investigation depends in no way upon the assumed theory of Cepheid variation.

The Base Line.—The distance of the globular cluster Messier 5 has been determined by various photometric methods.[2] I have derived the parallax $\pi = 0.000082''$ from investigations of the apparent diameter, the integrated apparent magnitude, the Harvard and Mount Wilson light curves of the variable stars, and the mean magnitude of the twenty-five brightest stars in the cluster. The estimated probable error is 10%,

which includes all recognized uncertainties, except the zero point of the period-luminosity curve of the Cepheid variables.

A recent discussion, by Dr. R. E. Wilson, of the proper motions of many more Cepheid variables than were available for my original determination of the zero point of the period-luminosity curve, shows that the earlier determination possesses no serious error.[3] There is, in fact, no reason as yet for altering the original value (in the sense of smaller distances for the clusters), when account is taken of the apparently necessary systematic correction to the Boss proper motions.

The adopted parallax of Messier 5 corresponds to a distance of approximately 40,000 light years. It is, of course, much greater than the base lines used heretofore, when, with the aid of eclipsing binaries, observations in light of different composition were made by Nordmann, Shapley, Russell, and others. The eclipsing stars have given diverse and uncertain results, generally because the phenomena observed are not capable of accurate timing, or because the observations in blue and yellow light were not simultaneous.

The Observations.—Five special series of photographs of Messier 5 were made by the writer in 1917, using the 60-inch reflector of the Mount Wilson Observatory. The exposures of twenty to thirty minutes necessary to record the yellow light with an isochromatic plate and yellow filter were, as a rule, interrupted in the middle for the exposure, one or two minutes in duration, of an ordinary plate to blue light. In this manner the variable stars were photographed in two regions of the spectrum at essentially identical times. The blue and yellow (photographic and photovisual) observations were ordinarily carried on throughout several hours of a night. Each night's work gave fragments of the light curves of all the variables; but, naturally, for only a few stars in each series was the light rising from minimum through median to maximum light, as the duration of a night's "run" was much less than the average period of variation.

The photographs were measured at the Mount Wilson Observatory, and the reductions mainly carried through at Harvard, using the customary method of deriving standard photographic magnitudes. Sixty-three hundred magnitudes were determined.

The Results.—The maximum effective wave-length of the blue light is approximately 4500 A, and of the yellow light, 5500 A. The observed differences in the times of rise through median magnitude, given in hundredths of a day in the fifth column of Table I, show that the velocities are sensibly equal; for the uncertainty of a measured difference is of the same order as the value derived. The differences Δt are photographic minus photovisual, a positive residual indicating that the yellow light is measured as arriving first.

For a single distant periodic signal it might be argued that the near

coincidence in the time of the arrival of blue and yellow light is not a positive indication that the velocities are the same, because the delay for one average wave-length with respect to another might be an integral multiple of the interval between signals. By using several variable stars, at the same distance but with slightly differing periods, as in Table I, this objection no longer has any weight.

The first and second columns of Table I contain the number assigned to the variables by Professor Bailey, and the first three significant figures in the periods determined by him. The amplitudes of variation in the

TABLE I

COMPARISON OF PHOTOGRAPHICALLY AND PHOTOVISUALLY MEASURED TIMES OF MEDIAN MAGNITUDE

NUMBER	PERIOD	RANGE PHOTOGRAPHIC	RANGE PHOTOVISUAL	Δt	WEIGHT
1	$0^d.522$	1.2	0.7	$+0^d.009$	3
				-0.001	2
4	0.450	1.4	0.9	-0.005	1
8	0.546	1.1	0.7	-0.012	1
12	0.468	1.3	1.0	-0.005	2
18	0.464	1.5	1.05	$+0.001$	1
20	0.610	0.9	0.7	0.000	2
				$+0.003$	1
28	0.544	1.2	0.8	$+0.006$	2
59	0.542	1.0	0.7	-0.003	1
63	0.500	1.2	0.9	-0.002	3
64	0.544	1.0	0.7	$+0.005$	1
81	0.557	1.1	0.8	$+0.001$	3
				-0.008	2

third and fourth columns are given to even tenths of a magnitude, as it is not possible from the Mount Wilson observations alone to determine them with more accuracy. The weight is assigned on the basis of the number of observations determining the rising branch of the light curve. Measures of the time of median magnitude were available on two different nights for three of the variables.

From Table I we derive the following mean result for the difference in time required for the passage of blue and yellow light over a distance of forty thousand light years:

$$\begin{aligned} \text{Blue—Yellow} &= -0^d.00012 \pm 0^d.0007 \\ &= -10 \text{ seconds} \pm 60 \text{ seconds} \end{aligned}$$

The probable error so far exceeds the observed mean difference that the latter has no numerical significance.

The error of the time measurement is, however, exceedingly small relative to the duration of time between the emission of the signal and its registration. The relative size of the probable error indicates that the

chances are even that the velocities of blue and yellow light do not differ by more than one part in twenty thousand million, notwithstanding the differences of nearly twenty-five per cent in wave-length. Probably they do not differ at all.

(There remains, however, the unlikely hypothesis that the blue light actually travels more rapidly in a vacuum, but that in interstellar space its speed is differentially diminished by absorbing matter to just the amount necessary to make it equal the observed speed of yellow light. Observations in another cluster would be a test of this assumption.)

From eleven determinations of the maxima for nine variables, a similar equality of velocities is found. In this case, also, the probable error of the result exceeds the observed average difference. The determination from the maxima, however, has much less weight, for the reason mentioned above, than the one based on the median magnitudes.

A graphical check on the preceding result is obtained by reducing the light curves of all the variables in Table I to composite curves, for blue and yellow light separately. Tables II and III contain the mean magnitudes for the composite light curves, which are plotted in the accompanying

TABLE II

COMPOSITE PHOTOGRAPHIC LIGHT CURVE—MEAN VALUES

PHASE	MAGNITUDE	PHASE	MAGNITUDE	PHASE	MAGNITUDE
$-0^d.108$	+0.46	$-0^d.033$	+0.51	$+0^d.025$	−0.48
−0.090	+0.50	−0.027	+0.48	+0.031	−0.53
−0.081	+0.53	−0.018	+0.38	+0.038	−0.55
−0.072	+0.52	−0.012	+0.20	+0.051	−0.48
−0.062	+0.56	−0.005	+0.10	+0.059	−0.44
−0.055	+0.48	+0.004	−0.10	+0.077	−0.31
−0.049	+0.53	+0.012	−0.23	+0.121	−0.13
−0.040	+0.54	+0.017	−0.42		

TABLE III

COMPOSITE PHOTOVISUAL LIGHT CURVE—MEAN VALUES

PHASE	MAGNITUDE	PHASE	MAGNITUDE	PHASE	MAGNITUDE
$-0^d.113$	+0.35	$-0^d.031$	+0.40	$+0^d.022$	−0.32
−0.083	+0.37	−0.022	+0.30	+0.037	−0.38
−0.065	+0.38	−0.005	+0.06	+0.057	−0.31
−0.051	+0.40	+0.013	−0.24	+0.095	−0.23

figure. The phases and the magnitudes for the individual stars have been reduced linearly to a mean period and range.[4] The coincidence of the photovisual and photographic curves at median and maximum magnitudes again illustrates the equality in the speed of light of these two different wave-lengths.

[1] Bailey, S. I., *Ann. Harv. Coll. Obs., Cambridge, Mass.*, **78**, 1917 (103–193).

[2] Shapley, Harlow, *Bull. Harv. Coll. Obs.*, No. **763**, 1922.

[3] Wilson, R. E., *Astron. J., Albany, N. Y.*, **35**, 1923 (35–44).

[4] Shapley, Harlow, *Circ. Harv. Coll. Obs.*, No. **237**, 1923 (1–11), p. 6.

SIZE-FACTORS AND SIZE-INHERITANCE

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In a previous article, published in these PROCEEDINGS[1] I referred in a footnote to a paper by Castle,[2] which had not come to my notice until the article in question had gone to press, but which dealt with certain problems that had largely occupied my own attention. Unfortunately, the position and wording of this footnote are such as to leave room for serious misinterpretation of my meaning. It might readily be inferred that I should not have arrived at the conclusions stated had I been aware of Castle's results. Indeed, one of my colleagues asked me if I really intended to "back-water" so whole-heartedly. Since it is my object here to deal critically with one of Castle's chief contentions, set forth in the paper mentioned, this has seemed a favorable opportunity to insist that I had no intention, in that footnote, of implying any abandonment of the views expressed in the text.

From the statistical treatment of a considerable series of rabbits, Castle concludes "that the genetic agencies affecting size in rabbits are general in their action, influencing in the same direction all parts of the body." At another point, he asserts the "strong probability" not only that the size-factors are "*general* in their action," but that they are "*exclusively* so." Furthermore, "the same would probably be found true for man and other mammals, perhaps for vertebrates in general."

Needless to say, Castle does not defend the impossible thesis that the growth of each of the various parts of the body is proportionate to that of the body as a whole. Indeed he discusses the changing proportions of certain regions of the human body with increase in stature, and concludes that racial or individual differences in respect to these proportions are due to factors which set a limit to the general process of growth. Large races of men have relatively longer limbs, small races relatively shorter ones. Thus, if the growth rates of the various members are conditioned by size-factors which "influence in the same direction all parts of the body," it must be that their specific local effects differ from one stage of ontogeny to another. This would, in itself, seem to be a rather far-reaching qualification of the original thesis. But, even as thus amended, I do not think that the hypothesis will stand.

Castle takes sharp issue with the contention of Davenport that size variations in different parts of the body may be independently inherited. On this general issue, I think there is no question that Davenport is correct.[3] Certain facts of comparative anatomy speak strongly for such inde-

pendent inheritance, aside from any special evidence which may be derived from biometrical studies.

Without comparing animals belonging to widely different taxonomic groups, we find within the single family of *Muridae* (the true mice and rats) striking differences in the proportional size of almost every bodily part. We have, for example, some species with a tail length which is less than a third that of the head and trunk combined; others with a tail length greatly exceeding that of the remainder of the body. In some of the *Microtinae* the ears are much less than half the length of the feet; in *Peromyscus truei truei* they are longer than the feet. Nor are these proportional differences in any way related to the absolute size of the body. Species with relatively long tails and ears are to be found among mice and rats representing both extremes in respect to size, while the short-tailed and short-eared genus *Microtus* contains both large and small species.

Moreover, we need not look to different genera or even to different species in order to find rather striking differences in the proportional size of body parts. In a number of previous papers,[4] I have given evidence of the independent variability of various parts of the body within the limits of a single species of *Peromyscus*. In comparing certain geographic races which occur in sequence, it is found that the tail and the foot may elongate together throughout a part of the range of the species, while elsewhere one of these members may become shorter or longer without involving the other. Again—to be specific, this time—the mean ear length of the Berkeley Hills race of *Peromyscus maniculatus* is distinctly less than that of the Mojave desert race (animals of equal size being compared), while in respect to both tail and foot length, the two races agree almost precisely. Such facts could without doubt be multiplied indefinitely by any taxonomic mammalogist or ornithologist who has given serious attention to the phenomena of geographic variation. That the differences under discussion are "genetic" ones has been sufficiently proved, at least for *Peromyscus*.

Furthermore, within practically every local race which I have studied in this regard, the males have, on the average, *longer* feet but *shorter* pelvic bones than the females, while the two sexes agree very closely in respect to tail length. These facts are of particular interest in the present connection, since, in both males and females, foot length and pelvis length are positively (*not* negatively) correlated.[5] In other words, within the same sex these parts vary together, while in the course of differentiation of the two sexes they have been modified in opposite directions.

As a consistent Mendelian, Castle would doubtless accept a factorial basis for all of these differences. But the action of some of these factors, if such really exist, must plainly be local and restricted rather than general, in their effects upon bodily growth.

Castle's extreme position in this matter appears to be based largely upon the very high correlations which he finds between the measurements for any two parts of the body in a certain series of rabbits. The coefficients computed by him range from +.741 (ear length and tibia length) to +.927 (femur length and tibia length).[6] In connection with these figures, certain things must be mentioned which greatly affect their significance for the purposes of the present discussion.

In the first place, the coefficients which he shows us represent *gross* correlations rather than *net* correlations. In any series of organisms which is heterogeneous in respect to size, a considerable correlation is bound to be encountered between almost any two characters measured. This is because the size of each character is largely dependent upon the size of the body as a whole. Large animals, for example, tend to have longer legs and longer ears than small animals, and for this reason a considerable degree of correlation between these two members is to be expected in a mixed population.

It may, of course, be replied that this, in itself, is evidence that the growth of the several members is conditioned by factors affecting the growth of the entire body. If by this is meant merely that the size attained by the various parts of the body is in a large measure dependent upon the size attained by the body as a whole, I should not think of disputing the proposition. Indeed it is hardly more than a truism. The question at issue is whether, despite this pronounced tendency for all of the parts to grow in unison, there is not a considerable margin of independent variability in respect to the size attained by any one of them.

Wright[7] has applied the method of partial correlation to some of Castle's own data, based upon measurements by MacDowell. He has computed the degree of correlation between two given parts (bones) which would exist were one or more of the other parts of a constant size. The gross coefficients obtained by Castle were in all cases diminished, though in no case reduced to zero. Even with three of the characters regarded as constant, positive correlations between each pair of remaining characters were obtained, these ranging from +.004 (obviously non-significant) to +.517. Since rabbits having three measurements constant "should be nearly free from variations due to general size factors," Wright concludes that in addition to variations due to these last, "there is a certain amount of variation of each bone length independently of all others measured and there are also groups of bones which vary together independently of the rest of the body."

In the course of our studies of *Peromyscus*, we have computed coefficients representing the correlations between a large proportion of the characters which are capable of quantitative expression.[8] In many of these cases we have computed both "gross" and "net" correlations, the

latter being (theoretically, at least) those which would be obtained were we dealing with animals of constant body length.[9]

The following table gives these figures, so far as they relate to measurements of *size*. This last distinction is, however, an arbitrary one, since differences in pigmentation are as truly quantitative as those here considered. Each of the coefficients given in the table is the mean of the coefficients for several local races, the two sexes likewise having been dealt with separately. For reasons which need not here be discussed, only "wild" (i.e., not cage-bred) animals have been included in the present computations. The first, second and sixth figures are based upon eight local races (over 900 individuals), the remaining figures being based upon four races (over 500 individuals).

It is, of course, impossible to give probable errors for these composite coefficients. Many, though not all, of the single figures thus averaged were of high statistical probability. It may be safely affirmed that all of the "gross" coefficients of the table are of statistical certainty, while the significance of the "net" ones is highly probable, with the exception of those for tail-pelvis and tail-skull, which are somewhat doubtful.

	GROSS σ	NET σ
Tail and foot	.411	.301
Tail and ear	.323	.130
Tail and pelvis	.500	.135
Tail and skull	.505	.153
Foot and pelvis	.349	.168
Foot and ear	.258	.157
	.391	.174

It will be noted, in the first place, that our "gross" coefficients indicate an average correlation less than half that found by Castle. The probable reason for this difference will be discussed shortly. In the second place, it is evident that while the "net" coefficients average less than half the size of the "gross" ones, they still point to the existence of an appreciable degree of correlation between the structures under consideration. In other words, not only do mice of the same size (body length) differ from one another in the length of various members, but these latter parts tend to vary together, independently of the body as a whole. This is in conformity with the findings of Wright, already referred to.

Another method of revealing the independent variability of certain parts has consisted in the computation of "net" standard deviations, i.e., ones from which is eliminated that portion of the total variability which is dependent upon the varying body size of the animals used. These "net" standard deviations are obtained by multiplying the "gross" figures by the factor $\sqrt{1-r^2}$, r being the coefficient of correlation between the character in question and body length. Such a corrected deviation repre-

sents the probable variability which the character in question would have in a series of constant body length.

The followng table gives the "gross" and "net" standard deviations for five measured characters of *Peromyscus*, all of these being characters which are more or less strongly influenced by the size of the body as a whole. Each figure is the mean of the figures for four separate local collections, the two sexes likewise being considered separately in the case of most characters. In the column headed "number" is given the number of individuals upon which each figure is based. The figures in the "percentage" column are the quotients obtained by dividing the "net" by the "gross" figures.

	NUMBER	GROSS σ	NET σ	PERCENTAGE
Foot length	516	.64	.59	92.2
Ear length	513	.79	.72	91.1
Pelvis length	332	.96	.61	63.5
Femur length	329	.82	.55	67.1
Skull length	333	.68	.48	70.6

It will be seen that in the case of these five characters the "net" variability ranges from nearly two-thirds to more than nine-tenths the magnitude of the gross. In other words, by far the greater part of the variation in the size of these structures is independent of variations in the size of the body as a whole.

Now it is of course pertinent to inquire at this point whether the variability here considered is of the genetic type. If these "net" figures (both correlation coefficients and standard deviations) merely represent a residuum of non-hereditary variations, they are obviously not relevant to any discussion of hereditary "size-factors."

That the variability here considered is, in part at least, of the hereditary type is shown by data at my disposal. Coefficients of parent-offspring correlation have been computed for one size-character which is largely independent of the total size of the body. I refer to the *relative* length of the tail, i.e., tail length regarded as a percentage of body length (head + trunk).[10] This percentage is, it is true, feebly correlated with the general size of the animal, larger individuals having relatively shorter tails. But the correlation is very low. It averages −.15 for series showing a considerable range of age, and would doubtless be much less if we restricted ourselves to fully mature animals. Furthermore, the "net" standard deviation for this character, as defined above (5.31), is 97.4 per cent of the "gross" figure (5.45).

Nevertheless, these variations in the relative length of the tail are in a considerable measure inherited, as I have pointed out in a prevoius paper.[11] The mean parent-offspring correlation, based upon 22 separate series, comprising 1449 individuals, is +.288. It therefore seems plain, for one

body part, at least, not only that it varies in size, quite independently of the general size of the body, but that these independent size variations are to some extent of the hereditary type. While I have not determined "coefficients of heredity" for the proportional size (ratio to body) of any other body parts, it would surely be surprising if similar results were not obtained for some of these.[12]

Returning to Castle's series of correlation coefficients, a second point to be borne in mind in considering their magnitude is the nature of the population upon which they are based. There are included, in Castle's total population, two small races and a large one, as well as two generations of hybrids derived from inter-breeding these. In such a case, it is obvious that one would obtain a high correlation between any two measured parts, provided that these varied in the same direction in the large and small races which were mixed, and that the mean differences between these races were great in proportion to the independent variations occurring within each single race. Now both of these conditions are fulfilled in the case of Castle's rabbits.[13]

On the other hand, it is equally obvious that one could obtain a quite different set of figures by throwing together races which did not fulfil the conditions stated. I am not familiar with the races of rabbits, but it is possible to select two subspecies of *Peromyscus* which yield very low or even negative correlations—at least for certain pairs of characters—if they are mixed together in a single statistical "population." Let us, for example, combine the large subspecies *P. maniculatus dubius*, having a tail of moderate length, with the medium sized subspecies *rubidus*, having a very long tail. Whereas, in the two races, taken singly, the gross correlation between foot length and tail length averages $+.449$, when these series are combined we have negative correlations for both sexes, 111 males giving $-.125 \pm .063$, and 102 females giving $-.265 \pm .062$.

Correlation coefficients, based upon mixed populations such as the foregoing, are the joint result of two different things, which should not be confused: (1) *intra-racial* variation, and (2) *inter-racial* variation. In more than one previous paper,[14] I have pointed out that two characters may be found to vary together when we pass from one geographic race to another, while they fail to display such a correlation within the limits of the single races. And the converse is likewise true.

According to the prevailing mendelian-mutation view, any such hereditary differences, whether between races or individuals, must be due to different combinations of unit factors, in the present case of "size-factors." Granting, for the moment, the truth of this conception, it would seem to be utterly impossible to explain certain of the phenomena which I have discussed on the assumption that these size factors are all "general in their action, influencing in the same direction all parts of the body." Whether,

in reality, the phenomena of size inheritance are satisfactorily accounted for in mendelian terms at all is another question, and one which it is not my object to discuss here.

Is it irrelevant for me to remark in conclusion that genetic theories must sooner or later reckon with the data of taxonomy and of geographic variation?

[1] THESE PROCEEDINGS, Feb., 1923.

[2] *Carnegie Inst. Publ.*, No. **320**, May, 1922.

[3] *Genetics*, July, 1917. My agreement with Davenport's main contention does not imply acceptance of his belief "that general factors control growth only to a degree that may be estimated as less than half."

[4] *Amer. Naturalist*, Apr.-May, 1918; May-June, 1923; *J. Exp. Zool.*, April, 1920.

[5] *J. Exp. Zoöl.*, Apr., 1920, p. 382. (This correlation is net as well as gross. The evidence now available is even stronger than when the foregoing paper was written.)

[6] Earlier correlations, based upon MacDowell's data (*Carnegie Inst. Publ.*, No. **196**, 1914) were materially lower than these, but were nevertheless high, owing to similar circumstances.

[7] Genetics, July, 1918.

[8] Some of these have been published in *J. Exp. Zoöl.*, Apr., 1920, and Oct., 1923.

[9] The method of "partial correlation" was here employed. (For formula, see Yule, *Theory of Statistics*, 1911, p. 247, *not* pp. 234, 235, where an astonishing typographical error occurs.) The computations were performed by Mr. R. R. Huestis.

[10] All of the measurements thus far considered have been absolute ones.

[11] *Amer. Naturalist*, June-July, 1918. Several additional series have been included in the computation of the coefficient here presented.

[12] Another relative value (width of tail stripe: circumference of tail), which is quite independent of body size, gives a mean "coefficient of heredity" very close to the last, but this character might not be regarded as relevant to a discussion of "size-factors." (I cannot, myself, see why not.)

[13] In one case, where Castle included only the two pure races, he obtained a coefficient of +.980 (femur and humerus).

[14] *J. Exp. Zoöl.*, Apr., 1920; *Amer. Naturalist*, May-June, 1923.

A GENERALIZATION OF VOLTERRA'S DERIVATIVE

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Volterra has given a definition for the derivative $\psi'(\mu p)$ at the place μp of a numerically valued function $\psi(\mu)$ whose argument μ is a real-valued continuous function of a real variable p on the interval (a, b). We show in the present paper how this definition may be modified in form so as to apply to the more general case where μ is a function belonging to a class of functions on a general range.

The Basis ($\mathfrak{P}\mathfrak{M}$). To this end let $\mathfrak{P}$ be a class $[p]$ of elements p. We

make no assumption concerning the class $\mathfrak{P}$ nor concerning its elements p. Let $\mathfrak{M}$ be a class $[\mu]$ of numerically valued functions μ each defined on $\mathfrak{P}$.

The Relation $R^{\mu p}$. For each μ in $\mathfrak{M}$ and p in $\mathfrak{P}$ there may be defined a binary relation $R^{\mu p}$ as follows: μ_1 is in the $R^{\mu p}$ relation to μ_2, in notation, $\mu_1 R^{\mu p} \mu_2$, if and only if

$$
\begin{aligned}
&(0) \quad \mu_1 \neq \mu;\\
&(1) \quad \mu_2(p) - \mu(p) \neq 0;\\
&(2) \quad [\mu_1(p') - \mu(p')][\mu_1(p') - \mu_2(p')] \leqq 0 \quad (p');\\
&(3) \quad [\mu(p') - \mu_2(p')][\mu(p'') - \mu_2(p'')] \geqq 0 \quad (p'p'').
\end{aligned}
$$

It is clear that for each μp the relation $R^{\mu p}$ is transitive and has the composition property as defined by E. H. Moore and the writer.[1]

The Limit $L^{\mu p}$.—By virtue of these properties the relation $R^{\mu p}$ serves to define a limit process $L^{\mu p}$. Thus the limit of $\psi(\mu')$, a numerically valued function on $\mathfrak{M}$, is a as μ' approaches μ at the place p, in notation, $L^{\mu p}_{\mu} \psi(\mu') = a$ or more briefly $L^{\mu p} \psi = a$, if there exists a system $(\mu_e \mid e)$ such that

$$| a - \psi(\mu') | \leqq e$$

for every μ' such that $\mu' R^{\mu p} \mu_e$.

The Derivative ψ'_{φ}.—Let us denote the difference quotient

$$\frac{\psi(\mu_1) - \psi(\mu_2)}{\varphi(\mu_1) - \varphi(\mu_2)}$$

by $\psi_{\varphi}(\mu_1 \mu_2)$. If $L^{\mu p}_{\mu} \psi_{\varphi}(\mu \mu')$ exists, we define it as the derivative of ψ with respect to φ at the place μp and denote it by $d\psi/d\varphi$ or more explicitly by $(d\psi/d\varphi)_{\mu p}$ or $\psi'_{\varphi}(\mu p)$.

Comparison with Volterra.—If $\mathfrak{P}$ is now taken to be the real interval (a, b) and $\mathfrak{M}$ the class of all continuous real-valued functions on (a, b), and φ is defined by the equation

$$\varphi(\mu) = \int_a^b \mu(p) dp$$

then $\psi'_{\varphi}(\mu p)$ becomes Volterra's derivative. The proof of this is simple and is omitted.

The above definition may be made the basis for a theory of functions of the type $\psi(\mu)$ above. Portions of such a theory have been in the writer's possession since the fall of 1920.

[1] Moore and Smith, "A General Theory of Limits," *Am. Journal of Math.*, XLIV (1922), p. 103.